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On the Properties of Time-Dependent, Force-Free, Degenerate Electrodynamics

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ABSTRACT

This paper formulates time-dependent, force-free, degenerate electrodynamics as a hyperbolic system of conservations laws. It is shown that this system has four characteristic modes, a pair of fast waves propagating with the speed of light and a pair of Alfvén waves. All these modes are linearly degenerate. The results of this analytic study can be used in developing upwind numerical schemes for the electrodynamics of black hole and pulsar magnetospheres. As an example, this paper describes a simple one-dimensional numerical scheme based on linear and exact Riemann solvers.

Key words: black hole physics – pulsars:general – magnetic fields – methods:numerical

1 INTRODUCTION

In magnetospheres of pulsars and black holes the electromagnetic field is so strong that the inertia and pressure of the plasma can be ignored. As a result the Lorentz force, $F_{\alpha\beta}J^{\alpha}$, vanishes and the transport of energy and momentum is entirely electromagnetic (Goldreich & Julian 1969; Blandford & Znajek 1977). This justifies the name "force-free" to describe the electrodynamics of pulsars and black holes. However, the electrodynamics of the magnetospheres is rather different from that in a vacuum which is obviously also force-free. Indeed, the magnetospheric plasma is plentiful enough to support strong electric currents and screen the electric field (Goldreich & Julian 1969; Blandford & Znajek 1977). An electromagnetic field satisfying this condition is called "degenerate" (Macdonald & Thorne 1982). Thus, the electrodynamics of pulsar and black hole magnetospheres is force-free, degenerate electrodynamics (FFDE).

So far, theoretical studies of pulsar and black hole magnetospheres have assumed a steady-state, indeed the properties of FFDE as a system of time-dependent equation have not been studied systematically. It is only recently that Uchida (1997) has developed a rather elegant theory of this system in which the electromagnetic field is described in terms of two scalar functions, called "Euler potentials". However, this formulation is not particularly suitable for numerical analysis because its basic equations, when written in components, involve mixed space and time second order derivatives. In this paper, we present another formulation in which the evolution equations have the form of hyperbolic conservation laws. We also describe a simple one-dimensional, upwind numerical scheme based on these analytic results. This scheme can easily be generalized to multidimensions and curved space-time. In fact, we have already constructed a 2D scheme for the Kerr space-time and used it (Komissarov 2001) to study the Blandford-Znajek mechanism (Blandford & Znajek 1977) for the extraction of rotational energy from black holes.

2 EVOLUTION EQUATIONS OF DEGENERATE ELECTRODYNAMICS

Since far away from the relativistic "star" the inertia of particles is bound to become important, the approximation of ideal relativistic magnetohydrodynamics (RMHD) could be considered as more satisfactory in general (Phinney1982). This suggests one way of deriving the time-dependent equations of FFDE, namely by considering the low inertia limit of RMDH. We shell adopt this approach because it reveals the close connection between these two systems. However, the time-dependent equations of FFDE can also be derived entirely within the frame of electrodynamics in a rather obvious way. Besides that, some of the microphysical conditions of MHD, like the small mean free path of plasma particles, do not have to be satisfied in FFDE.

In the covariant form the equations of RMHD are as follows (Lichnerowicz 1967; Anile 1989)

Maxwell's equation:

$$\nabla_\alpha {}^*\boldsymbol{F}^{\alpha\beta} = 0, \tag{1}$$

Energy-momentum equation:

$$\nabla_\alpha \boldsymbol{T}^{\alpha\beta} = 0, \tag{2}$$

Continuity equations

$$\nabla_\alpha (n_{(i)}\boldsymbol{u}^\alpha) = 0, \tag{3}$$

where $\boldsymbol{T}^{\alpha\beta}$ is the total stress-energy-momentum tensor, ${}^*\boldsymbol{F}^{\alpha\beta}$ is the dual tensor of electromagnetic field, $n_{(i)}$ is the proper number density for species i and $\boldsymbol{u}^\alpha$ is the fluid 4-velocity. The total stress-energy-momentum tensor

$$\boldsymbol{T}^{\alpha\beta} = \boldsymbol{T}^{\alpha\beta}_{(m)} + \boldsymbol{T}^{\alpha\beta}_{(f)}, \tag{4}$$

is the sum of the stress-energy-momentum tensor of matter, $\boldsymbol{T}^{\alpha\beta}_{(m)}$, and the stress-energy-momentum tensor of electromagnetic field

$$\boldsymbol{T}^{\alpha\beta}_{(f)} = \boldsymbol{F}^{\alpha}{}_{\mu}\boldsymbol{F}^{\beta\mu} - \frac{1}{4}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu})\boldsymbol{g}^{\alpha\beta}, \tag{5}$$

where $\boldsymbol{g}_{\alpha\beta}$ is the metric tensor and $\boldsymbol{F}_{\alpha\beta}$ is the electromagnetic field tensor. In this paper we assume that Greek indices run from 0 to 3 and Latin indices from 1 to 3.

These differential equations (2–3) are supplemented by equations of state and the perfect conductivity condition

$$\boldsymbol{F}_{\nu\mu}\boldsymbol{u}^\mu = 0. \tag{6}$$

This condition makes the second group of Maxwell equations redundant. From (6) one obtains

$$\boldsymbol{F} \cdot {}^*\boldsymbol{F} = 0, \tag{7}$$

$$\boldsymbol{F} \cdot \boldsymbol{F} > 0, \tag{8}$$

where $\boldsymbol{F} \cdot \boldsymbol{F}$ means $\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$. Just like classical MHD, RMHD is a hyperbolic system of conservation laws and has waves of four types – entropy, fast, Alfvén, and slow (e.g Anile 1989; Komissarov 1999).

If the contribution of matter to the total tensor of stress-energy momentum is very small then a perturbation technique suggests itself. The zero order equations constitute the system of FFDE:

$$\nabla_\mu {}^*\boldsymbol{F}^{\mu\nu} = 0, \tag{9}$$

$$\nabla_\mu \boldsymbol{T}^{\nu\mu}_{(f)} = 0, \tag{10}$$

From (5,9) one has

$$\nabla_\mu \boldsymbol{T}^{\nu\mu}_{(f)} = -\boldsymbol{F}_{\nu\mu}\boldsymbol{J}^\mu, \tag{11}$$

where

$$\boldsymbol{J}^\mu = \nabla_\nu \boldsymbol{F}^{\mu\nu}$$

and, thus, (10) requires the vanishing of the Lorentz force. Since equations (9,10) do not involve $\boldsymbol{u}_\alpha$, the perfect conductivity condition should be replaced with conditions (7,8), which express the degeneracy of electromagnetic field in FFDE. They ensure the existence of observers who detect only magnetic field. Let $\boldsymbol{a}^\mu$ to be the velocity of such observer. It easy to see that

$$\boldsymbol{F}_{\mu\nu}\boldsymbol{a}^\nu = 0$$

and, thus, $\boldsymbol{a}^\nu$ is a zero eigenvector of $\boldsymbol{F}_{\mu\nu}$. The second zero eigenvector can then be introduced via

$$\boldsymbol{b}^\mu = {}^*\boldsymbol{F}^{\nu\mu}\boldsymbol{a}_\nu.$$

In terms of $\boldsymbol{a}^\mu$ and $\boldsymbol{b}^\mu$

$$\boldsymbol{F}_{\mu\nu} = \boldsymbol{e}_{\mu\nu\alpha\beta}\boldsymbol{a}^\alpha\boldsymbol{b}^\beta, \tag{12}$$

and

$${}^*\boldsymbol{F}^{\mu\nu} = 2\boldsymbol{b}^{[\mu}\boldsymbol{a}^{\nu]}. \tag{13}$$

From the last equation it follows that the unit space-like vector $\boldsymbol{c}^\mu$ orthogonal to $\boldsymbol{a}^\mu$ and $\boldsymbol{b}^\mu$ is a zero eigenvector of ${}^*\boldsymbol{F}_{\mu\nu}$. The second zero eigenvector, also space-like and orthogonal to $\boldsymbol{a}^\mu$ and $\boldsymbol{b}^\mu$, is given by

$$\boldsymbol{d}^\mu = \boldsymbol{F}^{\nu\mu}\boldsymbol{c}_\nu.$$

In terms of these vectors

$$ {}^*\boldsymbol{F}_{\mu\nu} = \boldsymbol{e}_{\mu\nu\alpha\beta}\boldsymbol{c}^\alpha \boldsymbol{d}^\beta, \tag{14}$$

and

$$ \boldsymbol{F}^{\mu\nu} = 2\boldsymbol{c}^{[\mu}\boldsymbol{d}^{\nu]}. \tag{15}$$

From (7) it follows that there are only five independent components of $\boldsymbol{F}_{\mu\nu}$. In components, (9) splits into three evolution equations and one differential constraint on the initial data ($\nabla_i B^i = 0$). Thus, if the system (7)–(10) is self-consistent then (10) has only two independent components. Indeed, equations (11,12) show that that in the limit of degenerate electrodynamics, the vector $\nabla_\mu \boldsymbol{T}^{\nu\mu}$ always belongs to the two-dimensional vector space generated by $\boldsymbol{c}^\mu$ and $\boldsymbol{d}^\mu$.

The evolution equations of the system (9,10) have the form of conservation laws. In the following section we shall show that this system is also hyperbolic.

3 HYPERBOLICITY OF DEGENERATE ELECTRODYNAMICS

In order to demonstrate the hyperbolicity of FFDE, it is sufficient to study its one-dimensional equations in a locally pseudo-Cartesian system of coordinates. In such a frame, the electric and magnetic fields are defined via

$$ E_i = F_{i0}, \quad B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}, \tag{16}$$

where ϵ^{ijk} is the 3-dimensional Levi-Civita symbol. Then

$$ T^{00} = \frac{1}{2}(B^2 + E^2), \quad T^{i0} = \epsilon^{ijk}E_j B_k,$$

$$ T^{ij} = -(E^iE^j + B^iB^j) + \frac{\delta^{ij}}{2}(E^2 + B^2),$$

and conditions (7,8) read

$$ \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{B}} = 0, \tag{17}$$

$$ B^2 - E^2 > 0. \tag{18}$$

Notice that we use units such that the speed of light and π do not appear in the equations.

Like in MHD, the number of evolution equations and dependent variables for the one-dimensional system is reduced by one. Indeed, if $\partial/\partial x^2 = \partial/\partial x^3 = 0$ then x^0- (t-) and the x^1-components of (9) require

$$ B_1 = const.$$

Thus, the original system of FFDE is reduced to four independent evolution equations for four dependent variables, B_2, B_3, and two components of the electric field. The selection of these two components depends on the direction of magnetic field. For example, if $B_1 = 0$ then one of them must be E_1. As a result, we have to study three different sets of 1D equations to cover all possible cases. One can avoid such complications by relaxing the degeneracy condition (17) and modifying the equations in a way such that this condition is satisfied automatically if it is satisfied by the initial data. For example, one can select all three space components of the energy-momentum equation and add the term $B_i\partial(B_iE^i)/\partial t$ to obtain

$$ F_{i\mu}\frac{\partial F^{\mu t}}{\partial t} + F_{i\mu}\frac{\partial F^{\mu 1}}{\partial x^1} + B_i\frac{\partial (B_iE^i)}{\partial t} = 0. \tag{19}$$

Together with the two remaining components of (9) these equations constitute what we shell call the "augmented one-dimensional system of FFDE". It has five independent equations for five components of $\boldsymbol{F}_{\nu\mu}$. Obviously, solutions of this augmented system satisfying the constraint (17) are also solutions of the original system of FFDE. Moreover, contraction of (19) with B^i gives

$$ (B^iE_i)\frac{\partial E^1}{\partial x^1} + B^2\frac{\partial (B^iE_i)}{\partial t} = 0$$

This result shows that if $B^iE_i = 0$ at $t = 0$ then all time derivatives of B^iE_i at $t = 0$ also vanish and, therefore, $B^iE_i = 0$ at $t > 0$ as well.

All characteristic waves of the FFDE must be present in the augmented system because of the way it has been constructed. It is also expected to have one additional nonphysical wave due to higher number of evolution equations. This extra wave can easily be identified because across this wave E_iB^i does not have to stay constant.

In vector form the augmented system reads

$$\boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{C}\frac{\partial \boldsymbol{U}}{\partial x^1} = 0, \tag{20}$$

where

$$\boldsymbol{U} = (B_2, B_3, E_1, E_2, E_3)^t, \tag{21}$$

is the vector of dependent variables,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ B_1E_2 & B_1E_3 & B_1^2 & B_1B_2 - B_3 & B_1B_3 + B_2 \\ B_2E_2 & B_2E_3 & B_2B_1 + B_3 & B_2^2 & B_2B_3 - B_1 \\ B_3E_2 & B_3E_3 & B_3B_1 - B_2 & B_2B_3 + B_1 & B_3^2 \end{pmatrix}, \tag{22}$$

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -B_2 & -B_3 & E_1 & 0 & 0 \\ B_1 & 0 & E_2 & 0 & 0 \\ 0 & B_1 & E_3 & 0 & 0 \end{pmatrix}. \tag{23}$$

Given (17,18) the eigenvalue problem

$$(\boldsymbol{A} - \mu\boldsymbol{C}) \cdot \boldsymbol{r} = 0 \tag{24}$$

has the following solutions

$$\mu_f^{\pm} = \pm 1, \tag{25}$$

$$\mu_a^{\pm} = \frac{B_3E_2 - B_2E_3 \pm \sqrt{B_1^2(B^2 - E^2)}}{B^2}, \tag{26}$$

$$\mu_n = 0. \tag{27}$$

It is easy to verify that in the limit of FFDE the wave speeds of the fast waves of relativistic MHD (Anile 1989; Komissarov 1999) reduce to $\mu_f^{\pm}$ and the wave speeds of Alfvén waves of relativistic MHD reduce to $\mu_a^{\pm}$. Hence, μ_n corresponds to the nonphysical wave of the augmented system.

If $B_1 \neq 0$ then the eigenvectors of the fast modes are

$$\boldsymbol{r}_f^{\pm} = \left(-\eta_f,\ \zeta_f,\ 0,\ \mu_f^{\pm}\zeta_f,\ \mu_f^{\pm}\eta_f\right)^t, \tag{28}$$

where

$$\eta_f = E_3 + \mu_f^{\pm}B_2, \quad \zeta_f = E_2 - \mu_f^{\pm}B_3. \tag{29}$$

It is shown in Sec.4 that these waves have the same properties as linearly polarised electromagnetic waves in vacuum.

The eigenvectors of the Alfvén modes are

$$\boldsymbol{r}_a^{\pm} = \left(\zeta_a,\ \eta_a,\ -\frac{(\eta_a^2 + \zeta_a^2)}{B_1},\ \mu_a^{\pm}\eta_a,\ -\mu_a^{\pm}\zeta_a\right)^t, \tag{30}$$

where

$$\eta_a = E_3 + \mu_a^{\pm}B_2, \quad \zeta_a = E_2 - \mu_a^{\pm}B_3, \tag{31}$$

In some applications it might be useful to determine the left eigenvectors as well as the right eigenvectors. Unfortunately, the author was unable to find a concise form for the left eigenvectors and they are therefore not presented here.

Like MHD, the system of FFDE is not strictly hyperbolic, that is it allows multiple eigenvalues under certain conditions. Such degenerate⋆ cases require separate treatment because the eigenvectors given by (28) and (30) become either singular or linearly dependent.

(i) Provided

$$\vec{\boldsymbol{E}}_t = -\vec{\boldsymbol{i}}_1 \times \vec{\boldsymbol{B}}, \tag{32}$$

where $\vec{\boldsymbol{i}}_{\boldsymbol{1}}$ is the unit vector along the x_1-axis and $\vec{\boldsymbol{E}}_t$ is the tangential component of electric field, one has

⋆ In this paper three different kinds of degeneracy are encountered: 1) the degeneracy of electromagnetic field itself (see Sec.1), 2) the multiplicity of eigenvalues of the Jacobean matrix (Sec.3), and 3) the so-called linear degeneracy of hyperbolic waves (see Sec.4).

$$\mu_a^+ = \mu_f^+ = +1.$$

and the corresponding eigenvectors form the two-dimensional vector space generated by

$$(0,\ 1,\ 0,\ 1,\ 0)^t \quad \text{and} \quad (-1,\ 0,\ 0,\ 0,\ 1)^t. \tag{33}$$

Obviously, vacuum electromagnetic waves propagating in the positive direction have the same eigenspace. Thus, in this limit the right Alfvén wave also behaves like a linearly polarised electromagnetic wave in vacuum. Similarly, if

$$\vec{\boldsymbol{E}}_t = \vec{\boldsymbol{\imath}}_1 \times \vec{\boldsymbol{B}}, \tag{34}$$

one has

$$\mu_a^- = \mu_f^+ = -1,$$

and the corresponding eigenvectors form the two-dimensional vector space generated by

$$(0,\ 1,\ 0,\ -1,\ 0)^t \quad \text{and} \quad (1,\ 0,\ 0,\ 0,\ 1)^t. \tag{35}$$

This eigenspace is the same as that of vacuum electromagnetic waves propagating in the negative direction. We notice that conditions (32) and (34) cannot be satisfied simultaneously unless $\vec{\boldsymbol{E}} = 0$ and $\vec{\boldsymbol{B}}_t = 0$.

(ii) If $B_1 = 0$ then

$$\mu_a^+ = \mu_a^- = \mu_a = \frac{E_2}{B^3} = -\frac{E_3}{B_2} \tag{36}$$

with a two-dimensional vector space of eigenvectors. One can use

$$\boldsymbol{r} = (1,\ 0,\ \frac{-B_2(\mu_a^2 - 1)}{E_1},\ 0,\ -\mu_a)^t \tag{37}$$

and

$$\boldsymbol{r} = (0,\ 1,\ \frac{-B_3(\mu_a^2 - 1)}{E_1},\ \mu_a,\ 0)^t \tag{38}$$

as the basis of the eigenspace if $E_1 \neq 0$ or

$$\boldsymbol{r} = (0,\ 0,\ 1,\ 0,\ 0)^t \tag{39}$$

and

$$\boldsymbol{r} = (B_3,\ -B_2,\ 0,\ -\mu_a B_2,\ -\mu_a B_3)^t \tag{40}$$

if $E_1 = 0$.

4 PROPERTIES OF WAVES OF DEGENERATE ELECTRODYNAMICS

4.1 Fast waves

From (28) one obtains the following system of differential equations describing the fast simple wave:

$$\frac{dB_2}{-(E_3 + \mu_f B_2)} = \frac{dB_3}{E_2 - \mu_f B_3} = \frac{dE_1}{0} = \frac{dE_2}{\mu_f(E_2 - \mu_f B_3)} = \frac{dE_3}{\mu_f(E_3 + \mu_f B_2)}, \tag{41}$$

where μ_f is either $\mu_f^+ = +1$ or $\mu_f^- = -1$. These are easily integrated to obtain

$$E_1 = \text{const.} \tag{42}$$

$$\eta_f = E_3 + \mu_f B_2 = \text{const}, \tag{43}$$

$$\zeta_f = E_2 - \mu_f B_3 = \text{const}, \tag{44}$$

$$E_i B^i = \text{const}, \tag{45}$$

If we introduce the vector

$$\vec{\boldsymbol{t}}_f = (0, \zeta_f, \eta_f)^t$$

then

$$d\vec{\boldsymbol{E}}_t \parallel \vec{\boldsymbol{t}}_f \quad \text{and} \quad d\vec{\boldsymbol{B}}_t \perp \vec{\boldsymbol{t}}_f, \tag{46}$$

where $\vec{\boldsymbol{B}}_t$ and $\vec{\boldsymbol{E}}_t$ are the tangential components of the fields. Thus, these are transverse waves and they have the same properties as linearly polarised electromagnetic waves in vacuum.

4.2 Alfvén waves

From (30) one obtains the equations of an Alfvén simple wave:

$$\frac{dB_2}{E_2-\mu_a B_3} = \frac{dB_3}{E_3+\mu_a B_2} = -\frac{B_1 dE_1}{\eta_a^2+\zeta_a^2} = \frac{dE_2}{\mu_a(E_3+\mu_a B_2)} = -\frac{dE_3}{\mu_a(E_2-\mu_a B_3)}, \tag{47}$$

where μ_a is either μ_a^+ or μ_a^-. First, we show that

$$B^2-E^2 = \text{const} \tag{48}$$

and

$$\mu_a = \text{const}. \tag{49}$$

Since both these properties are Lorentz invariant it is sufficient to show that they hold in one particular frame. In this case, the most convenient frame is the "fluid" frame, where $\vec{\boldsymbol{E}}=\vec{\boldsymbol{0}}$ and, hence, $dE^2=0$. Moreover, (47) ensures

$$d(B^2) = d(B_3E_2 - B_2E_3) = 0$$

and the results (48,49) are manifest.

Next, equations (47) can easily be integrated to obtain

$$\eta_a = E_3+\mu_a B_2 = \text{const}, \tag{50}$$

$$\zeta_a = E_2-\mu_a B_3 = \text{const}, \tag{51}$$

$$\zeta_a E_2+\eta_a E_3 = \text{const}, \tag{52}$$

$$E_i B^i = \text{const}, \tag{53}$$

Since the normal component of electric field varies the Alfvén waves of FFDE are not quite transverse. This is similar to what has been found for Alfvén waves in relativistic MHD (Komissarov 1997). These waves can still be described as linearly polarised. If we introduce the vector

$$\vec{\boldsymbol{t}}_a = (0, \zeta_a, \eta_a)^t$$

then

$$d\vec{\boldsymbol{E}}_t \perp \vec{\boldsymbol{t}}_a \quad \text{and} \quad d\vec{\boldsymbol{B}}_t \parallel \vec{\boldsymbol{t}}_a. \tag{54}$$

From this equation and (46) one can see that the polarisations of Alfvén and electromagnetic waves propagating in the same direction are not, in general, mutually orthogonal.

If $B_1=0$ then from (26) it follows that Alfvén waves propagate with the drift velocity

$$\mu_a = (\vec{E}\times\vec{B})_1/B^2,$$

and, therefore, must be somewhat similar to the contact wave of relativistic MHD. Their properties are most transparent in the wave frame, where $\mu=0$. From (36) one can see that in such frame

$$\vec{\boldsymbol{E}}_t = \vec{\boldsymbol{0}}. \tag{55}$$

Using this result and (37) one finds that

$$(-B_1,\ B_2,\ 0,\ 0,\ 0)^t \quad \text{and} \quad (E_1,\ 0,\ B_2,\ 0,\ 0)^t \tag{56}$$

is a basis of the Alfvén eigenspace. The first vector describes the familiar rotation of $\vec{\boldsymbol{B}}_t$. The second allows variation of the magnitude of $\vec{\boldsymbol{B}}_t$, but requires

$$B_t^2 - E_1^2 = \text{const}.$$

Equations (25,49) show that all waves of FFDE are linearly degenerate, that is

$$\boldsymbol{r}\cdot\boldsymbol{\nabla}_U\mu = 0,$$

and, therefore, this system does not allow the formation of shocks via wave steepening. However, discontinuities may be introduced via discontinuous initial data and/or boundary conditions. For linearly degenerate waves the shock equations coincide with jump equations of simple waves (Boillat 1982; Anile 1989).

Finally, figure 1 shows the normal wave speed diagram in the "fluid" frame, where $\vec{E}=\vec{0}$. Since in this frame

$$\mu_a^{\pm} = \pm\cos\theta,$$

where θ is the angle between $\vec{\boldsymbol{B}}$ and the wave vector, all curves on this diagram are circles.

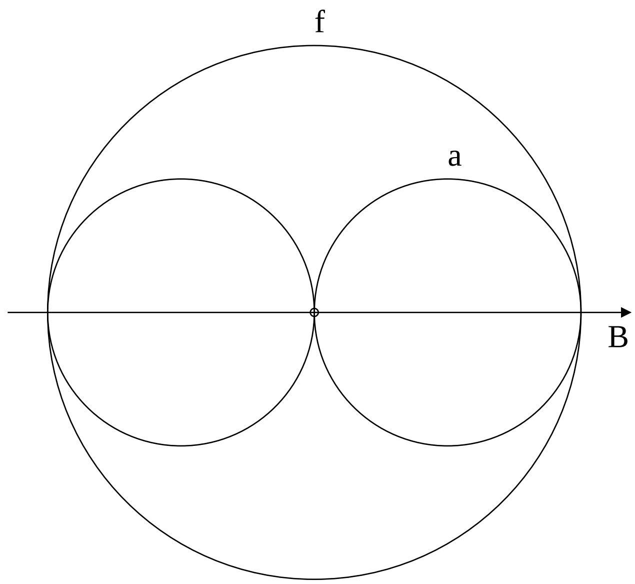

Figure 1. The normal speed diagram in the "fluid" frame ($\vec{\boldsymbol{E}} = \vec{\boldsymbol{0}}$).

5 1D UPWIND NUMERICAL SCHEME

Nowadays, the construction of upwind numerical schemes for hyperbolic conservation laws is a well established area of numerical analysis. Here we describe a simple one-dimensional scheme which is very similar to the one constructed by Komissarov (1999) for RMHD. The space-time is considered to be flat.

5.1 General Structure

The one-dimensional conservation laws of FFDE can be written in the form

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial x^1} = 0, \tag{57}$$

where the vector of conserved variables is

$$\boldsymbol{Q} = (S_1, S_2, S_3, B_2, B_3)^t,$$

where $\vec{\boldsymbol{S}}$ is the Poynting flux vector, and the vector of hyperbolic fluxes is

$$\boldsymbol{P} = (T_{11}, T_{12}, T_{13}, -E_3, E_2)^t.$$

Notice that we use all three space components of the energy-momentum equation. In addition, we introduce the auxiliary vector of "primitive" variables

$$\boldsymbol{U} = (B_2, B_3, E_1, E_2, E_3)^t.$$

These primitive variables can be converted into the conservative ones and vice versa via

$$\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}, \tag{58}$$

$$\vec{\boldsymbol{E}} = \frac{1}{B^2}\vec{\boldsymbol{S}} \times \vec{\boldsymbol{B}}. \tag{59}$$

The second equation automatically ensures the degeneracy condition (17). Unfortunately, this does not mean that the second condition, (18), will also be satisfied and extra care would have to be taken in cases where $B^2 - E^2$ is close to zero. At the moment we do not have the means to ensure (17).

Let us define a regular Cartesian grid such that the ith cell is centered at $x_i = ih$ and occupies the region between $x_{i-1/2} = (i - 1/2)h$ and $x_{i+1/2} = (i + 1/2)h$, where h is the mesh spacing. Now suppose that we know the solution at

$t = t_n$ and we want to calculate it at a later time t_{n+1}. We integrate equations (57) over the ith cell and from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$ to get

$$\boldsymbol{Q}_{i,n+1} - \boldsymbol{Q}_{i,n} + \frac{\Delta t}{h}(\boldsymbol{P}_{i+\frac{1}{2},n+\frac{1}{2}} - \boldsymbol{P}_{i-\frac{1}{2},n+\frac{1}{2}}) = 0. \tag{60}$$

Here

$$\boldsymbol{Q}_{i,n} = \frac{1}{h}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \boldsymbol{Q}(x,t_n)dx$$

is the mean value of $\boldsymbol{Q}$ in the ith cell at time t_n and

$$\boldsymbol{P}_{i+\frac{1}{2},n+\frac{1}{2}} = \frac{1}{\Delta t}\int\limits_{t_n}^{t_{n+1}} \boldsymbol{P}(x_{i+\frac{1}{2}},t)dt,$$

is the time average of the flux at the cell interface with $x = x_i + \frac{1}{2}h$.

In a first order Godunov-type scheme (Godunov 1959), it is assumed that at the beginning of each time step the solution is uniform within each cell. This implies initial discontinuities at the cell interfaces and the time averaged fluxes can therefore be found by solving the corresponding Riemann problems. In fact, the solution, $\boldsymbol{U}^*_{i+1/2}$, of the problem at the interface with $x = x_{i+1/2}$, does not depend on time but only on the initial left, $\boldsymbol{U}_{i,n}$, and right, $\boldsymbol{U}_{i+1,n}$, states. Then the first order fluxes can be computed from

$$\boldsymbol{P}_{i+\frac{1}{2},n+\frac{1}{2}} = \boldsymbol{P}(\boldsymbol{U}^*_{i+\frac{1}{2}}).$$

In order to achieve second order accuracy, we use the first order scheme to obtain the solution, $\boldsymbol{Q}_{i,n+\frac{1}{2}}$ and hence $\boldsymbol{U}_{i,n+\frac{1}{2}}$ at the half time, $t_{n+\frac{1}{2}} = t_n + \Delta t/2$. We then use this to compute average gradients of primitive variables in each cell as follows

$$\left(\frac{\partial \boldsymbol{U}}{\partial x}\right)_{i,n+\frac{1}{2}} = \frac{1}{h}\mathrm{av}\left(\Delta\boldsymbol{U}_{i,n+\frac{1}{2}}, \Delta\boldsymbol{U}_{i+1,n+\frac{1}{2}}\right),$$

where

$$\Delta\boldsymbol{U}_{i,n+\frac{1}{2}} = \left(\boldsymbol{U}_{i,n+\frac{1}{2}} - \boldsymbol{U}_{i-1,n+\frac{1}{2}}\right),$$

and $\mathrm{av}(a,b)$ is a non-linear averaging function whose purpose is to reduce the scheme to first order in space in the neighborhood of discontinuities. Here we will adopt the same averaging function as in (Falle 1991):

$$\mathrm{av}(a,b) = \begin{cases} \dfrac{(a^2b + ab^2)}{(a^2+b^2)} & \text{if } ab \geq 0, \text{ and } a^2+b^2 \neq 0, \\[2ex] 0 & \text{if } ab < 0 \text{ or } a^2+b^2 = 0, \end{cases}$$

These gradients can now be used to set up the left and right states for the second order Riemann problems

$$\boldsymbol{U}^l_{i+1/2} = \boldsymbol{U}_{i,n+\frac{1}{2}} + \frac{1}{2}h\left(\frac{\partial \boldsymbol{U}}{\partial x}\right)_{i,n+\frac{1}{2}},$$

$$\boldsymbol{U}^r_{i+1/2} = \boldsymbol{U}_{i+1,n+\frac{1}{2}} - \frac{1}{2}h\left(\frac{\partial \boldsymbol{U}}{\partial x}\right)_{i+1,n+\frac{1}{2}}.$$

The solution, $\boldsymbol{U}^*_{i+1/2} = \boldsymbol{U}(\boldsymbol{U}^l_{i+1/2}, \boldsymbol{U}^r_{i+1/2})$, to this Riemann problem allows us to compute the second order fluxes

$$\boldsymbol{P}_{i+\frac{1}{2},n+\frac{1}{2}} = \boldsymbol{P}(\boldsymbol{U}^*_{i+\frac{1}{2}}),$$

which are used to advance the solution through the full time step from $t = t_n$ to $t = t_{n+1}$ according to (60).

5.2 Riemann Solvers

By a Riemann problem we mean the initial value problem for (57) with the initial conditions

$$\boldsymbol{U}(x,0) = \begin{cases} \boldsymbol{U}^l & \text{for} \quad x \leq 0, \\ \boldsymbol{U}^r & \text{for} \quad x > 0. \end{cases} \tag{61}$$

For a general hyperbolic system this problem has a self-similar solution which involves only shocks and centered rarefactions (e.g. Landau & Lifshitz 1959, Jeffrey & Taniuti 1964). Since all waves of FFDE are linearly degenerate, centered simple waves do not exist and, thus, the solution involves only discontinuities.

5.2.1 Linear Riemann Solver

Since all waves are linearly degenerate we have

$$\boldsymbol{U}_r = \boldsymbol{U}_l + \sum_{i=1,4} \mathcal{L}_i \boldsymbol{r}_i, \tag{62}$$

where $\boldsymbol{r}_i$ is the eigenvector of the upstream state of ith wave. For fast waves these states are already known, they are the left and the right states of the Riemann problem. Upstream states of Alfvén waves are different and depend on the amplitudes of fast waves. In our linear Riemann solver we ignore this difference. Then (62) becomes a linear system of five equations for four unknowns, $\mathcal{L}_i$. To handle it, we ignore one of the equations. The corresponding four components of the solution at $x^1 = 0$ are then found via

$$\boldsymbol{U}^* = \boldsymbol{U}_l + \sum_{\mu_i<0} \mathcal{L}_i \boldsymbol{r}_i = \boldsymbol{U}_r - \sum_{\mu_i>0} \mathcal{L}_i \boldsymbol{r}_i. \tag{63}$$

The remaining component is determined using the degeneracy condition (17).

5.2.2 Exact Riemann Solver

In most Riemann problems originated from numerical simulations the difference between left and right states is rather small and linear Riemann solvers are sufficiently accurate. However, for handling strong discontinuities one might require an exact solution of the Riemann problem. Thanks to the simplicity of FFDE, an exact Riemann solver can be constructed rather easily.

First we consider the case where none of the states of the Riemann problem has multiple eigenvalues. From the results of Sec.4 it follows that only fast waves change the value of $p = B^2 - E^2$ and only Alfvén waves change the value of E_1. This implies that the Riemann problem reduces to determining the values of p and E_1 in the region bounded by Alfvén waves, $\bar{p}$ and $\bar{E}_1$. Given the left state, the values of E_1 and p in the right state, and $\bar{p}$ and $\bar{E}_1$ the corresponding right state $\boldsymbol{V}_r(\bar{p}, \bar{E}_1)$ can be found by successive application of the jump equations from left to right:

$$\frac{[B_2]}{-\eta_f} = \frac{[B_3]}{\zeta_f} = \frac{[E_1]}{0} = \frac{[E_2]}{\mu_f \zeta_f} = \frac{[E_3]}{\mu_f \eta_f} = \frac{[p]}{-2\mu_f(\eta_f^2 + \zeta_f^2)}, \tag{64}$$

for fast waves, and

$$\frac{[B_2]}{\zeta_a} = \frac{[B_3]}{\eta_a} = -\frac{B_1[E_1]}{\eta_a^2 + \zeta_a^2} = \frac{[E_2]}{\mu_a \eta_a} = -\frac{[E_3]}{\mu_a \zeta_a} = \frac{[p]}{0} \tag{65}$$

for Alfvén waves. This gives us the system of nonlinear equations

$$\boldsymbol{V}_r(\bar{p}, \bar{E}_1) = \boldsymbol{U}_r. \tag{66}$$

which can then be solved iteratively. Once $\bar{p}$ and $\bar{E}_1$ are found the resolved state is determined via (64,65).

In degenerate cases the problem is even easier.

(i) If $\mu_f^+ = \mu_a^+$ in the right state of the Riemann problem, then the right fast and Alfvén waves merge into a single degenerate wave across which both p and E_1 are constant Thus we immediately obtain $\bar{p} = p_r$, $\bar{E}_1 = E_{1_r}$ and then determine the resolved state in the same way as in the non-degenerate case. Similarly, if $\mu_f^- = \mu_a^-$ in the left state then $\bar{p} = p_l$, $\bar{E}_1 = E_{1_l}$.

(ii) It is easy to verify that for any of the degenerate waves, the corresponding degeneracy condition holds on both sides of the wave. Thus, if μ_f^+ equals to μ_a^+ in the right state and μ_f^- equals to μ_a^- in the left state then in the resolved state conditions (32) and (34) are satisfied simultaneously. This immediately gives $\vec{\boldsymbol{E}} = 0$ and $\vec{\boldsymbol{B}}_t = 0$ in the resolved state.

(iii) If $B_1 = 0$ then the Alfvén waves merge into one contact-like wave across which both p and μ_a are constant. This leads to the following two equations:

$$p_l^*(\mathcal{L}_{f_l}) = p_r^*(\mathcal{L}_{f_r}), \tag{67}$$

$$\mu_{a_l}^*(\mathcal{L}_{f_l}) = \mu_{a_r}^*(\mathcal{L}_{f_r}), \tag{68}$$

where * indicates states adjacent to the contact wave. Simple manipulations reduce this system to a quadratic equation for $\mathcal{L}_{e_l}$ (or $\mathcal{L}_{e_r}$). It appears that only one of its solutions satisfies the condition (18).

5.3 Test calculations

We have tested the scheme against exact analytic solutions for plane electromagnetic and Alfvén waves including all degenerate cases. The exact solutions can easily be constructed using the results of Sec.3 and 4. The profiles of these waves should remain

unchanged as they propagate. Figures 2–3 illustrate the results for some of the test calculations. In these figures the initial and the final exact solutions are shown by continuous lines and markers show the final numerical solution.

Fast wave: Figure 2a shows the results for a fast wave propagating in the positive direction. The initial solution is

$$B_1 = 1.0, \quad B_3 = E_2 = 0.0,$$

$$B_2 = \begin{cases} 1.0 & \text{if} \quad x^1 < -0.1 \\ 1.0 + \frac{3}{2}(x^1 + 0.1) & \text{if} \quad -0.1 < x^1 < 0.1 \\ 1.3 & \text{if} \quad x^1 > 0.1 \end{cases}$$

Alfvén wave: Figure 2b shows the results for a non-degenerate Alfvén wave propagating in the negative direction with the wave speed $\mu = -0.5$. The initial solution is

$$B'_1 = B'_2 = 1.0, \quad E'_2 = E'_3 = 0.0,$$

$$B'_3 = \begin{cases} 1.0 & \text{if} \quad x^1 < -0.1 \\ 1.0 + \frac{3}{2}(x^1 + 0.1) & \text{if} \quad -0.1 < x^1 < 0.1 \\ 1.3 & \text{if} \quad x^1 > 0.1 \end{cases},$$

where $\vec{B}'$ and $\vec{E}'$ are measured in the wave frame which is moving relative to the grid with speed $\mu = -0.5$.

Degenerate Alfvén wave: Figure (3) shows the results for a rotational degenerate Alfvén wave propagating to the right with $\mu = 0.5$. The initial solution is

$$\vec{E}' = 0, \quad B'_1 = 0,$$

$$B'_2 = 2\cos\phi, \quad B'_3 = 2\sin\phi,$$

where

$$\phi = \begin{cases} 0.0 & \text{if} \quad x^1 < -0.1 \\ \frac{5\pi}{2}(x^1 + 0.1) & \text{if} \quad -0.1 < x^1 < 0.1 \\ \frac{\pi}{2} & \text{if} \quad x^1 > 0.1 \end{cases}.$$

Three waves problem: In this last test the initial discontinuity at $x = 0$ splits into two fast discontinuities and a stationary Alfvén discontinuity. The initial solution is

$$\begin{array}{lll} \vec{\boldsymbol{B}} = (1.0, 1.5, 3.5) & \vec{\boldsymbol{E}} = (-1.0, -0.5, 0.5) & \text{if} \quad x^1 < 0, \\ \vec{\boldsymbol{B}} = (1.0, 2.0, 2.(3)) & \vec{\boldsymbol{E}} = (-1.5, 1.(3), -0.5) & \text{if} \quad x^1 > 0. \end{array}$$

As one can see, numerical diffusion smoothes out the sharp wave fronts of the exact solutions but otherwise the agreement is very good indeed.

Not all Riemann problems with both the left and the right states satisfying the degeneracy conditions (17,18) have solutions within the frame of degenerate electrodynamics. For example, the following problem does not seem have one

$$\begin{array}{lll} \vec{\boldsymbol{B}} = (1.0, 1.0, 1.0) & \vec{\boldsymbol{E}} = (0.0, 0.5, -0.5) & \text{if} \quad x^1 < 0, \\ \vec{\boldsymbol{B}} = (1.0, -1.0, -1.0) & \vec{\boldsymbol{E}} = (0.0, 0.5, -0.5) & \text{if} \quad x^1 > 0, \end{array}$$

as iterations of our exact Riemann solver do not converge. To make sure that this is not due to a fault in the solver we have considered a related problem where this initial discontinuity is substituted with a linear transition layer. Numerical simulations (see figure 5) show that within this layer $B^2 - E^2$ decreases in time toward zero leading to the breakdown of the condition

$$B^2 - E^2 > 0.$$

As $B^2 - E^2$ tends to zero the drift velocity of plasma

$$\vec{\boldsymbol{v}}_d = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}/B^2$$

tends to the speed of light suggesting that one may no longer neglect the inertia of plasma particles.

6 CONCLUSIONS

We have shown that the evolution equations of FFDE can be written in the form of conservation laws. One group of equations arise from the energy-momentum conservation. There are only two linear independent dynamic equations (components of one tensor equation) in this group. This is in contrast to MHD, where all four components of the energy-momentum equation are linearly independent, and vacuum electrodynamics, where in general $\boldsymbol{F}_{\mu\nu}$ does not have zero eigenvectors and the energy-momentum equation is reduced to

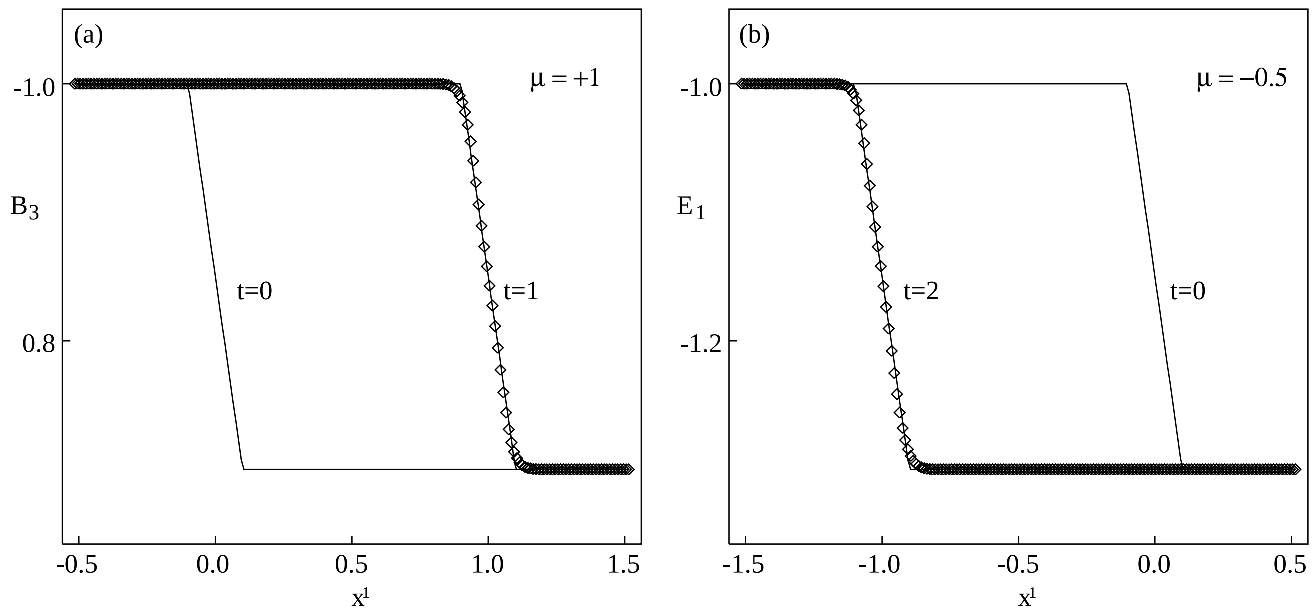

Figure 2. (a) The propagation of a fast wave. (b) The propagation of a non-degenerate Alfvén wave. The initial and the final exact solutions are shown by continuous lines and markers show the final numerical solution.

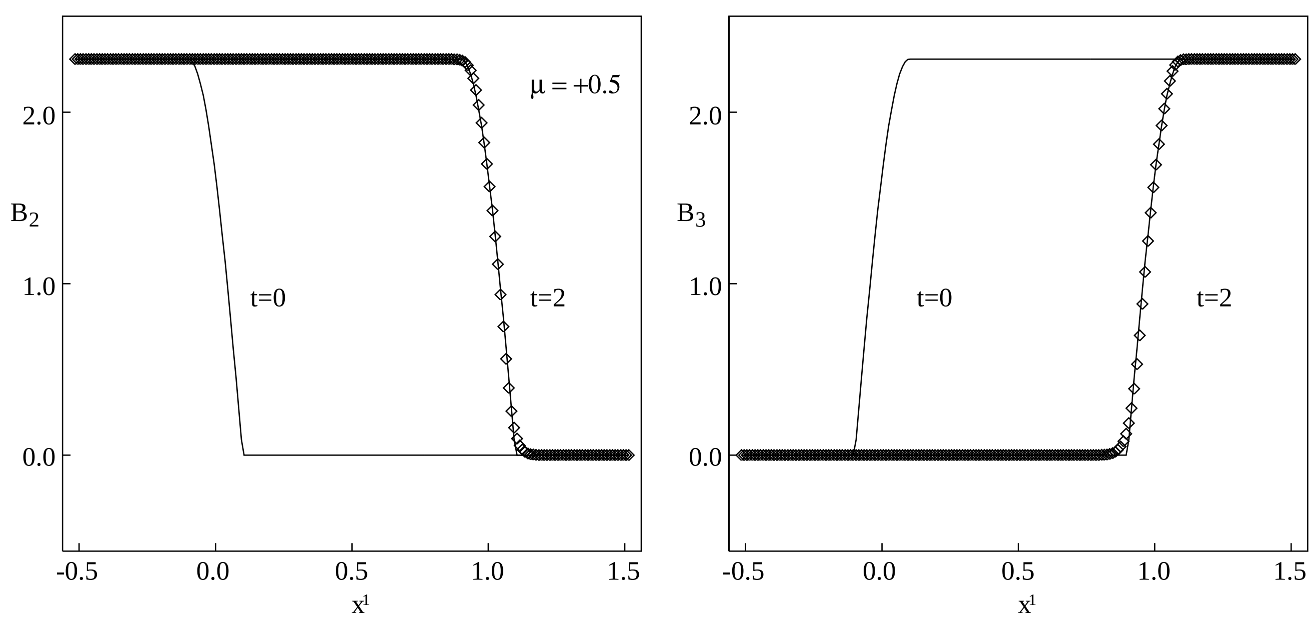

Figure 3. The propagation of a rotational degenerate Alfvén wave. The initial and the final exact solutions are shown by continuous lines and markers show the final numerical solution.

$\nabla_\nu F^{\mu\nu} = 0,$

which has three independent dynamic components. The one-dimensional system of FFDE with plane symmetry, in flat space-time, includes four linearly independent dynamic equations for four independent components of the electromagnetic field tensor (other equations simply require B_1 =const.) This system is hyperbolic, though not strictly hyperbolic, with a pair of fast waves and a pair of Alfvén waves. All characteristic modes are linearly degenerate, which means that this system does not allow the formation of shocks via steepening of continuous waves.

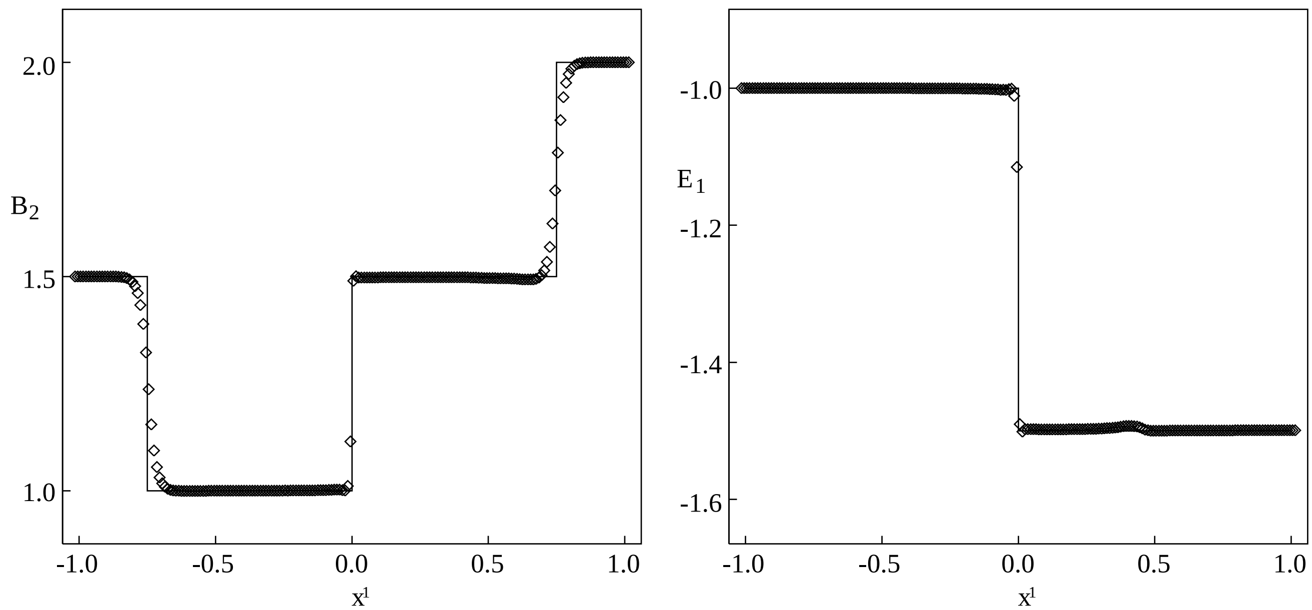

Figure 4. In this test problem the initial discontinuity at $x^1 = 0$ splits into two fast waves and one stationary Alfvén wave. The exact solution at $t = 0.75$ is shown by continuous lines and markers show the corresponding numerical solution.

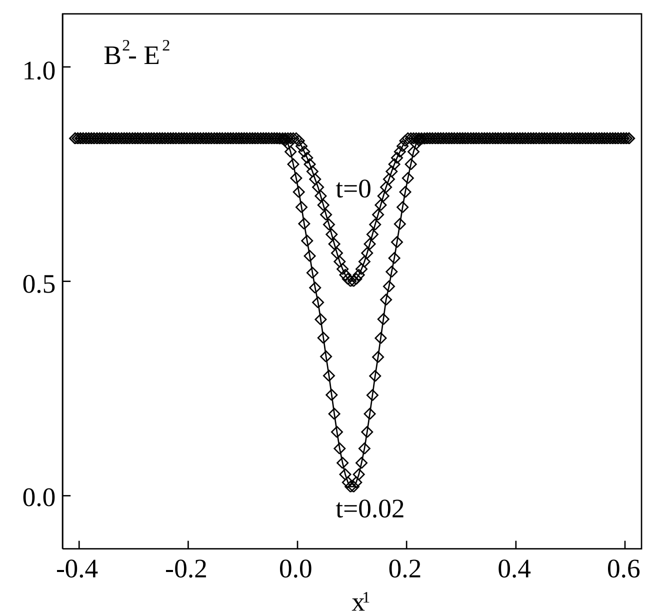

Figure 5. Breakdown of the approximation of degenerate electrodynamics as $B^2 - E^2 \to 0$. See the last problem described in Sec.5.4 .

This formulation is particularly useful when it comes to developing numerical schemes for time-dependent FFDE as there has been a great deal of work on numerical methods for hyperbolic conservation laws. Here we have presented a simple one-dimensional scheme based mainly on our linear Riemann solver. Its generalization to multi-dimensions is relatively straightforward. We have already constructed a 2D numerical scheme adapted to Kerr space-time which will be described elsewhere. Such schemes can be useful tools in studying electromagnetically driven jets and winds from black holes and neutron stars and, perhaps, other related phenomena of relativistic astrophysics.

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